A COMPLETE SOLUTION TO AN OPEN PROBLEM RELATING TO AN INEQUALITY FOR RATIOS OF GAMMA FUNCTIONS

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Abstract. In this paper, we prove that for $x+y>0$ and $y+1>0$ the inequality

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}<\sqrt{\frac{x+y}{x+y+1}}$$

is valid if $x>1$ and reversed if $x<1$, where $\Gamma(x)$ is the Euler gamma function. This completely extends the result in [Y. Yu, *An inequality for ratios of gamma functions*, J. Math. Anal. Appl. **352** (2009), no. 2, 967–970.] and thoroughly resolves an open problem posed in [B.-N. Guo and F. Qi, *Inequalities and monotonicity for the ratio of gamma functions*, Taiwanese J. Math. **7** (2003), no. 2, 239–247.].

1. Introduction

It is common knowledge that the classical Euler gamma function $\Gamma(x)$ may be defined for $x>0$ by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t. \tag{1}$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k\in\mathbb{N}$ are called the polygamma functions. It is general knowledge that these functions are basic and that they have much extensive applications in mathematical sciences.

In [6, Theorem 2] and its preprint [29], the function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{2}$$

was proved to be decreasing with respect to $x\ge1$ for fixed $y\ge0$. Consequently, the inequality

$$\frac{x+y+1}{x+y+2}\le\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}} \tag{3}$$

holds for positive real numbers $x\ge1$ and $y\ge0$.

Meanwhile, influenced by an inequality in [17] and its preprint [16], an open problem was posed in [6] and its preprint [29] to ask for an upper bound $\sqrt{\frac{x+y}{x+y+1}}$ for the function in the right-hand side of the inequality (3). Hereafter, such an open problem was repeated and modified in several papers such as [4, 5, 7, 18, 19, 20, 24, 25, 27].

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In [36], the above-mentioned open problem was affirmatively but partially resolved: If $y > 0$ and $x > 1$, then

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}} < \sqrt{\frac{x+y}{x+y+1}}\,; \tag{4}$$

if $y > 0$ and $0 < x < 1$, then the inequality (4) is reversed.

The main aim of this paper is to completely extend the one-side inequality (4) and to thoroughly resolves the open problem mentioned above.

Our main results may be recited as follows.

Theorem 1. *For $x + y > 0$ and $y + 1 > 0$, the inequality (4) holds if $x > 1$ and reverses if $x < 1$. The cases $x = 0, -1$ are understood to be the limits as $x \to 0, -1$ on both sides of the inequality (4), that is,*

$$\psi(y+1) > \frac{\ln y + \ln(y+1)}{2}, \quad y > 0 \tag{5}$$

and

$$\psi(y+1) < \frac{3\ln y - \ln(y-1)}{2}, \quad y > 1. \tag{6}$$

As a by-product of the proof of Theorem 1, we conclude the following inequality.

Corollary 1. *For $x + y > 0$ and $y + 1 > 0$, if $|x| < 1$, then*

$$\left[\frac{\Gamma(x+y+1)}{\Gamma(y+1)}\right]^{1/x} > \frac{(x+y)^{(x+1)/2}}{(x+y+1)^{(x-1)/2}}; \tag{7}$$

if $|x| > 1$, then the inequality (7) is reversed.

Remark 1. It is noted that necessary and sufficient conditions for the function (2) and its generalization to be logarithmically completely monotonic have been gained in [28] and related references therein.

Remark 2. Taking $y = 0$ and $x = \frac{n}{2}$ in Theorem 1 leads to

$$\frac{[\Gamma(n/2+1)]^{1/n}}{[\Gamma((n+2)/2+1)]^{1/(n+2)}} = \frac{\Omega_{n+2}^{1/(n+2)}}{\Omega_n^{1/n}} < \sqrt[4]{\frac{n}{n+2}} \tag{8}$$

for $n > 2$, where

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)} \tag{9}$$

stands for the n-dimensional volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. Similarly, if letting $y = 1$ and $x = \frac{n+1}{2} > 1$ in Theorem 1, then

$$\frac{\Omega_{n+5}^{1/(n+3)}}{\Omega_{n+3}^{1/(n+1)}} < \frac{1}{\pi^{2/(n+1)(n+3)}} \sqrt[4]{\frac{n+3}{n+5}}, \quad n \ge 2. \tag{10}$$

For more information on inequalities for the volume of the unit ball in $\mathbb{R}^n$, please see [1, 2, 26] and related references therein.

2. Lemmas

In order to prove Theorem 1, we need the following lemmas.

Lemma 1. *For $t > s > 0$ and $k \in \mathbb{N}$, we have*

$$\min\biggl\{s, \frac{s+t-1}{2}\biggr\} < \biggl[\frac{\Gamma(s)}{\Gamma(t)}\biggr]^{1/(s-t)} < \max\biggl\{s, \frac{s+t-1}{2}\biggr\} \tag{11}$$

and

$$\frac{(k-1)!}{\bigl(\max\bigl\{s, \frac{s+t-1}{2}\bigr\}\bigr)^k} < \frac{(-1)^{k-1}\bigl[\psi^{(k-1)}(t)-\psi^{(k-1)}(s)\bigr]}{t-s} < \frac{(k-1)!}{\bigl(\min\bigl\{s, \frac{s+t-1}{2}\bigr\}\bigr)^k}, \tag{12}$$

where $\psi^{(0)}(x)$ stands for $\psi(x)$. Moreover, the lower and upper bounds in (11) and (12) are the best possible.

Proof. For real numbers a, b and c, denote $\rho = \min\{a, b, c\}$, and let

$$H_{a,b;c}(x) = (x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{13}$$

with respect to $x \in (-\min\{a, b, c\}, \infty)$. In [30, 31], it was obtained that

(1) the function $H_{a,b;c}(x)$ is logarithmically completely monotonic, that is,

$$0 \le (-1)^i[\ln H_{a,b;c}(x)]^{(i)} < \infty$$

for $i \ge 1$, on $(-\rho, \infty)$ if and only if

$$\begin{aligned}(a, b; c) \in D_1(a, b; c) \triangleq \{&(a, b; c) : (b-a)(1-a-b+2c) \ge 0\}\\ &\cap \{(a, b; c) : (b-a)(|a-b|-a-b+2c) \ge 0\}\\ &\setminus \{(a, b; c) : a = c+1 = b+1\}\\ &\setminus \{(a, b; c) : b = c+1 = a+1\};\end{aligned} \tag{14}$$

(2) so is the function $H_{b,a;c}(x)$ on $(-\rho, \infty)$ if and only if

$$\begin{aligned}(a, b; c) \in D_2(a, b; c) \triangleq \{&(a, b; c) : (b-a)(1-a-b+2c) \le 0\}\\ &\cap \{(a, b; c) : (b-a)(|a-b|-a-b+2c) \le 0\}\\ &\setminus \{(a, b; c) : b = c+1 = a+1\}\\ &\setminus \{(a, b; c) : a = c+1 = b+1\}.\end{aligned} \tag{15}$$

In [35], the classical asymptotic relation

$$\lim_{x\to\infty}\frac{\Gamma(x+s)}{x^s\Gamma(x)} = 1 \tag{16}$$

for real s and x was confirmed. This relation implies that

$$\lim_{x\to\infty} H_{a,b;c}(x) = 1. \tag{17}$$

From the logarithmically complete monotonicity of $H_{a,b;c}(x)$, it is deduced that the function $H_{a,b;c}(x)$ is decreasing if $(a, b; c) \in D_1(a, b; c)$ and increasing if $(a, b; c) \in D_2(a, b; c)$ on $(-\rho, \infty)$. As a result of the limit (17) and the monotonicity of the function $H_{a,b;c}(x)$, it follows that the inequality $H_{a,b;c}(x) > 1$ holds if $(a, b; c) \in D_1(a, b; c)$ and reverses if $(a, b; c) \in D_2(a, b; c)$, that is, the inequality

$$x + \lambda < \biggl[\frac{\Gamma(x+a)}{\Gamma(x+b)}\biggr]^{1/(a-b)} < x + \mu$$

for $b > a$ holds if $\lambda \le \min\{a, \frac{a+b-1}{2}\}$ and $\mu \ge \max\{a, \frac{a+b-1}{2}\}$, which may be reduced to the inequality (11) by replacing $x+a$ and $x+b$ by s and t respectively.

Further, by virtue of the logarithmically complete monotonicity of $H_{a,b;c}(x)$ on $(-\rho,\infty)$ again and the fact [32, p. 82] that a completely monotonic function which is non-identically zero cannot vanish at any point on $(0,\infty)$, it is readily deduced that

$$\begin{aligned}(-1)^k[\ln H_{a,b;c}(x)]^{(k)} &= (-1)^k[(b-a)\ln(x+c)+\ln\Gamma(x+a)-\ln\Gamma(x+b)]^{(k)}\\ &= (-1)^k\left[\frac{(-1)^{k-1}(k-1)!(b-a)}{(x+c)^k}+\psi^{(k-1)}(x+a)-\psi^{(k-1)}(x+b)\right]\\ &> 0\end{aligned}$$

for $k \in \mathbb{N}$ is valid if $(a,b;c) \in D_1(a,b;c)$ and reversed if $(a,b;c) \in D_2(a,b;c)$. Consequently, the double inequality

$$-\frac{(k-1)!(b-a)}{(x+c_2)^k} < (-1)^k[\psi^{(k-1)}(x+b)-\psi^{(k-1)}(x+a)] < -\frac{(k-1)!(b-a)}{(x+c_1)^k}$$

holds with respect to $x \in (-\rho,\infty)$ if $(a,b;c_1) \in D_1(a,b;c)$ and $(a,b;c_2) \in D_2(a,b;c)$, which may be rearranged as

$$\frac{(k-1)!}{(x+\alpha)^k} < \frac{(-1)^{k-1}\left[\psi^{(k-1)}(x+b)-\psi^{(k-1)}(x+a)\right]}{b-a} < \frac{(k-1)!}{(x+\beta)^k} \tag{18}$$

for $x \in (-\rho,\infty)$ if $\alpha \ge \max\{a, \frac{a+b-1}{2}\}$ and $\beta \le \min\{a, \frac{a+b-1}{2}\}$, where $b > a$ and $k \in \mathbb{N}$. In the end, replacing $x+a$ and $x+b$ by s and t respectively in (18) leads to (12). The proof of Lemma 1 is thus complete. □

Remark 3. The double inequalities (11), (12) and (18) slightly extend the double inequalities in [8, Theorem 4.2] and [30, Theorem 3] which were not proved in detail therein.

Remark 4. For more information on the logarithmically complete monotonicity of the function (13), please refer to [8, 12, 13, 30, 31], especially the expository and survey papers [14, 15], and related references therein.

Lemma 2. *For $x \in (0,\infty)$ and $k \in \mathbb{N}$, we have*

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x} \tag{19}$$

and

$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1}\psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}}. \tag{20}$$

Proof. In [10, Theorem 2.1], [21, Lemma 1.3] and [22, Lemma 3], the function $\psi(x) - \ln x + \frac{\alpha}{x}$ was proved to be completely monotonic on $(0,\infty)$, i.e.,

$$(-1)^i\left[\psi(x)-\ln x+\frac{\alpha}{x}\right]^{(i)} \ge 0 \tag{21}$$

for $i \ge 0$, if and only if $\alpha \ge 1$, so is its negative, i.e., the inequality (21) is reversed, if and only if $\alpha \le \frac{1}{2}$. In [3, Theorem 2], [9, Theorem 2.1] and [11, Theorem 2.1], the function $\frac{e^x\Gamma(x)}{x^{x-\alpha}}$ was proved to be logarithmically completely monotonic on $(0,\infty)$, i.e.,

$$(-1)^k\left[\ln\frac{e^x\Gamma(x)}{x^{x-\alpha}}\right]^{(k)} \ge 0 \tag{22}$$

for $k \in \mathbb{N}$, if and only if $\alpha \ge 1$, so is its reciprocal, i.e., the inequality (22) is reversed, if and only if $\alpha \le \frac{1}{2}$. Considering the fact [32, p. 82] that a completely monotonic function which is non-identically zero cannot vanish at any point on $(0, \infty)$ and rearranging either (21) or (22) leads to the double inequalities (19) and (20). Lemma 2 is proved. $\square$

Lemma 3 ([23, 33, 34]). *If $t > 0$, then*

$$\frac{2t}{2+t} < \ln(1+t) < \frac{t(2+t)}{2(1+t)}; \tag{23}$$

If $-1 < t < 0$, the inequality (23) is reversed.

3. Proofs of Theorem 1 and Corollary 1

Now we are in a position to prove Theorem 1 and Corollary 1.

Proof of Theorem 1. When $0 \ge y > -1$ and $x > -y$, let

$$f_y(x) = \frac{\ln\Gamma(x+y+1) - \ln\Gamma(y+1)}{x} - \frac{1}{2}\ln(x+y); \tag{24}$$

When $y > 0$ and $x > -y$, define

$$f_y(0) = \psi(y+1) - \frac{1}{2}\ln y$$

and $f_y(x)$ for $x \ne 0$ to be the same one as in (24). Making use of the well-known recursion formula $\Gamma(x+1) = x\Gamma(x)$ and computing straightforwardly yields

$$\begin{aligned} f_y(x+1) - f_y(x) &= \left(\frac{1}{x+1} - \frac{1}{x}\right)\ln\frac{\Gamma(x+y+1)}{\Gamma(y+1)} \\ &\quad + \frac{\ln(x+y+1)}{x+1} + \frac{1}{2}\ln\frac{x+y}{x+y+1} \\ &= \frac{1}{x+1}\left\{\ln\left[\frac{(x+y)^{(x+1)/2}}{(x+y+1)^{(x-1)/2}}\right] - \ln\left[\frac{\Gamma(x+y+1)}{\Gamma(y+1)}\right]^{1/x}\right\}. \end{aligned} \tag{25}$$

Substituting $s = y+1 > 0$ and $t = x+y+1 > 1$ into (11) in Lemma 1 leads to

$$\min\left\{y+1, \frac{x+2y+1}{2}\right\} < \left[\frac{\Gamma(x+y+1)}{\Gamma(y+1)}\right]^{1/x} < \max\left\{y+1, \frac{x+2y+1}{2}\right\}$$

which is equivalent to

$$\left[\frac{\Gamma(x+y+1)}{\Gamma(y+1)}\right]^{1/x} < \begin{cases} \dfrac{x+2y+1}{2}, & x > 1 \\ y+1, & x < 1 \end{cases}$$

and

$$\left[\frac{\Gamma(x+y+1)}{\Gamma(y+1)}\right]^{1/x} > \begin{cases} y+1, & x > 1 \\ \dfrac{x+2y+1}{2}, & x < 1 \end{cases}$$

for $y+1 > 0$ and $x+y > 0$. Consequently, it follows readily from (25) that, for $y > -1$ and $x + y > 0$,

(1) if $x>1$ and
$$\frac{(x+y)^{(x+1)/2}}{(x+y+1)^{(x-1)/2}}>\frac{x+2y+1}{2}, \tag{26}$$
then $f_y(x+1)-f_y(x)>0$;

(2) if $-1<x<1$ and the inequality (26) reverses, then $f_y(x+1)-f_y(x)<0$.

For $x+y>0$ and $y>-1$, let
$$g(x,y)=\frac{(x+y)^{x+1}}{(x+2y+1)^2(x+y+1)^{x-1}}.$$
The partial differentiation of $g(x,y)$ with respect to y is
$$\frac{\partial g(x,y)}{\partial y}=\frac{1-x^2}{(x+2y+1)^3}\biggl(\frac{x+y}{x+y+1}\biggr)^x.$$
This shows that

(1) when $|x|>1$, the function $g(x,y)$ is strictly decreasing with respect to $y>-1$;

(2) when $|x|<1$, the function $g(x,y)$ is strictly increasing with respect to $y>-1$.

In addition, it is clear that $\lim_{y\to\infty}g(x,y)=\frac14$. As a result, it is easy to see that $g(x,y)\gtrless\frac14$ when $|x|\gtrless1$ for $x+y>0$ and $y>-1$. In other words, the inequality (26) is valid when $|x|>1$ and reversed when $|x|<1$ for all $x+y>0$ and $y>-1$. Consequently, the inequality $f_y(x+1)-f_y(x)>0$ holds if $x>1$ and reverses if $|x|<1$, where $x+y>0$ and $y>-1$.

For $x<-1$, denote the function enclosed in the braces in (25) by $Q(x,y)$. Direct computation yields
$$\begin{aligned}
Q(x,y)&=\frac{x+1}{2}\ln(x+y)-\frac{x-1}{2}\ln(x+y+1)-\frac1x\int_{y+1}^{x+y+1}\psi(u)\,\mathrm{d}u\\
&=\frac{x+1}{2}\ln(x+y)-\frac{x-1}{2}\ln(x+y+1)-\int_0^1\psi((y+1)(1-u)+(x+y+1)u)\,\mathrm{d}u
\end{aligned}$$
and
$$\begin{aligned}
\frac{\partial Q(x,y)}{\partial x}&=\frac{3x+2y+1}{2(x+y)(x+y+1)}+\frac12\ln\frac{x+y}{x+y+1}\\
&\quad-\int_0^1u\psi'((y+1)(1-u)+(x+y+1)u)\,\mathrm{d}u.
\end{aligned}$$
Making use of the left-hand side inequality for $k=1$ in (20) results in
$$\begin{aligned}
\frac{\partial Q(x,y)}{\partial x}&<\frac{3x+2y+1}{2(x+y)(x+y+1)}+\frac12\ln\frac{x+y}{x+y+1}\\
&\quad-\int_0^1u\biggl\{\frac{1}{(y+1)(1-u)+(x+y+1)u}+\frac{1}{2[(y+1)(1-u)+(x+y+1)u]^2}\biggr\}\,\mathrm{d}u\\
&=\frac12\biggl[\frac{x^2-2yx-y(2y+1)}{x(x+y)(x+y+1)}+\ln\frac{x+y}{x+y+1}-\frac{1+2y}{x^2}\ln\frac{y+1}{x+y+1}\biggr].
\end{aligned}$$
Further employing the left-hand side inequality of (23) in Lemma 3 leads to
$$\frac{\partial Q(x,y)}{\partial x}<\frac12\biggl[\frac{x^2-2yx-y(2y+1)}{x(x+y)(x+y+1)}-\frac{2}{1+2x+2y}+\frac{1+2y}{x^2}\cdot\frac{2x}{2+x+2y}\biggr]$$

$$
\begin{aligned}
&= \frac{(2y+3)x^2 + 2\big(y^2+2y+2\big)x + 3y + 2}{2(x+y)(x+y+1)(x+2y+2)(2x+2y+1)} \\
&\triangleq \frac{(2y+3)F_1(x,y)F_2(x,y)}{2(x+y)(x+y+1)(x+2y+2)(2x+2y+1)},
\end{aligned}
$$

where

$$F_1(x,y) = \left(x + \frac{2+y^2+2y-\sqrt{y^4+4y^3+2y^2-5y-2}}{2y+3}\right)$$

and

$$F_2(x,y) = \left(x + \frac{2+y^2+2y+\sqrt{y^4+4y^3+2y^2-5y-2}}{2y+3}\right).$$

For $x<-1$, $x+y>0$ and $y+1>0$, standard argument reveals that

$$F_1(x,y) < \frac{2+y^2+2y-\sqrt{y^4+4y^3+2y^2-5y-2}}{2y+3} - 1 < 0$$

and

$$F_2(x,y) > \frac{2+y^2+2y+\sqrt{y^4+4y^3+2y^2-5y-2}}{2y+3} - y > 0,$$

so $\frac{\partial Q(x,y)}{\partial x} < 0$ and the function $Q(x,y)$ is decreasing with respect to $x<-1$. From the fact that $Q(-1,y)=0$, it follows that $Q(x,y)>0$ for $x<-1$. Theorem 1 is thus proved. □

Proof of Corollary 1. This follows readily from the discussion in the proof of Theorem 1 about the positivity and negativity of the function enclosed by braces in (25). □

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